The roots of Russell's paradox

by Gregory H. Moore

AMONG LOGICIANS AND mathematicians, Russell is best known for Russell's Paradox and his way out of that paradox, the theory of types. Yet there has been surprisingly little work on the origins of his paradox. The usual account presupposes that Russell's Paradox arose from two earlier paradoxes—the paradox of the largest ordinal, due to Burali-Forti [1897], and the paradox of the largest cardinal, due to Cantor in 1897 but unpublished. In fact, however, neither Burali-Forti nor Cantor regarded himself as having discovered any paradox (i.e., an argument requiring some basic premiss of logic or mathematics to be abandoned). No one published a paradox in this sense until Russell did so in 1903. On this point the evidence is clear, though not well known (see Moore and Garciadiego 1981).

The present paper[1] argues that the roots of Russell's Paradox go deeper. They do not begin with his discovery of the paradox of the largest cardinal, but can be traced back to his numerous attempts to resolve the antinomy of infinite number. Less directly, the roots of Russell's Paradox grew in the Kantian and Hegelian soil in which Russell was educated at Cambridge. Both of those philosophical traditions relied heavily on antinomies, and Russell utilized such antinomies many times in his writings on the foundations of mathematics from 1896 onward.

Russell's use of antinomies up to 1898, when he ceased being a Hegelian, predisposed him to seek antinomies in logic even in 1900 when he became a follower of Peano. During such a search he found the paradox of the largest cardinal, which emerged—quite gradually—from his antinomy of infinite number and—more immediately—from his criticism of Cantor's theorem that the class of all subclasses of a class K has a larger cardinal than K. From the paradox of the largest cardinal he then extracted Russell's Paradox.

[1] This paper is a synopsis of that given at the conference, to be published in full in *The Journal of Symbolic Logic*. Much of the evidence given there has, for reasons of space, been omitted here.

Kant and Hegel

As a Cambridge undergraduate, Russell absorbed the German idealism of Kant and Hegel from his teachers James Ward and G.F. Stout, but his use of antinomies owes much to his reading of the Oxford idealist F.H. Bradley. "In 1894," Russell later wrote, "I went over completely to a semi-Kantian, semi-Hegelian metaphysic" (1959, p. 38).

Both Kant and Hegel emphasized antinomies. Four antinomies were a central feature of Kant's *Critique of Pure Reason*. Variations on them, particularly the second and third, occurred repeatedly in Russell's work. Kant's second antinomy stated that every composite substance both is, and is not, composed of simple parts. His third stated: there are two kinds of causality, one that of the laws of nature and the other that of freedom; there is only one kind of causality, that of the laws of nature.

In Hegel, contradictions played a more subtle role: showing the need for a higher synthesis. Any field of knowledge, such as geometry, necessarily included contradictions, which could be resolved only at a higher level of knowledge, e.g. physics. Later, Russell acknowledged that in 1898 he was

> a full-fledged Hegelian, and I aimed at constructing a complete dialectic of the sciences, which should end up with the proof that all reality is mental. I accepted the Hegelian view that none of the sciences is quite true, since all depend upon some abstraction, and every abstraction leads, sooner or later, to contradictions. Wherever Kant and Hegel were in conflict, I sided with Hegel. (1959, p. 42)

Russell's predilection for antinomies in a Kantian style is quite visible in his *Essay on the Foundations of Geometry* (1897). Its final chapter is devoted to the three contradictions he regarded as central to geometry. The first was the antinomy of the point: "Though the parts of space are intuitively distinguished, no conception is adequate to differentiate them" (1897, p. 188). The second contradiction concerned the relativity of position, while the third was that space consists merely of relations and yet of more than relations (1897, p. 198). He resolved these geometric antinomies in Hegelian fashion by appealing to a higher science: physics. These antinomies were criticized by several philosophers, notably Edward Dixon (1898, p. 8) and G.E. Moore (1899).

Russell discussed the antinomy of the point in his paper "Are Euclid's Axioms Empirical?": "Magnitude is created by comparison, even though the terms compared existed, of course, prior to the comparison. They become quantities only by virtue of quantitative comparison.... This is the contradiction of relativity" (1898, p. 763). He added that this contradiction, "a measurable difference defined between two terms that are nevertheless intrinsically exactly similar", was "what I understood by my antinomy of the point.... This contradiction penetrates all of mathematics" (1898, p. 764).

In 1896, after completing his fellowship dissertation on geometry, Russell developed a neo-Hegelian philosophy of physics. It is hard to characterize adequately the numerous antinomies which Russell formulated during this period, especially in "Various Notes on Mathematical Philosophy". In the note "Dynamics and Absolute Motion" an antinomy resulted from the fact that "for dynamics, it is *geometrically* necessary that our axes should be material and *dynamically* necessary that they should be *immaterial*.... This antinomy ... is so fundamental as to render a purely dynamical universe absurd...." In "Note on the Conception of a Plenum" he asked: "But if we allow matter to be a plenum, do not all the antinomies of space reassert themselves?" Finally, in "On Quantity and Allied Conceptions: An Enquiry into the Subject-Matter of Mathematics", he described in sweeping terms the contradictions that permeated mathematics:

> It would appear that one or other of the contradictions [in his "Note on Quantity and Quality"] applies everywhere except in Arithmetic: the qualitative contradiction applies to the subject-matter of the logical Calculus, and the quantitative contradiction everywhere else. The unique position of Arithmetic may perhaps disappear on analysis.

Russell learns of Cantor

In 1896 Russell reviewed a book on atomism by Arthur Hannequin. The term "atomism" referred not only to physical atoms but also to mathematical points as the ultimate constituents of space—in particular, to Cantor's attempt to base the linear continuum on the real numbers. "The fundamental proposition of the ... book," wrote Russell (1896a, p. 410), "is this: That all atomism results from the attempt to apply to continua the discrete conception of number, the atom being the discontinuous element required for numeration. Hence arise at once the necessity and the contradictions of atomism." Russell accepted this neo-Kantian position with its extreme ambivalence toward the modern foundations of mathematical analysis.

Thus, when he first encountered Cantorian set theory in Hannequin's book (1895), Russell reacted with mistrust. He rejected as impossible Cantor's attempt to explain the notion of continuum by means of real numbers, and observed:

> This impossibility leads Hannequin to the first fundamental contradiction of atomism, the necessary divisibility of the indivisible element. This is only our old friend, Kant's second antinomy, but it acquires a new force by the proof of its inherence in mathematical method. (1896a, p. 412)

Moreover, Russell rejected Cantor's infinite ordinal numbers since even the *finite* ordinal numbers can never come to an end. These two themes—Cantor's treatment of the continuum and the possible existence of an infinite number—were intimately involved with the emergence of the paradox of the largest cardinal and thus, at one remove, with the discovery of Russell's Paradox.

Stimulated by Hannequin's book, Russell read Cantor. He then examined the nature of a continuum, and the antinomies to which it gave rise, in "On Some Difficulties of Continuous Quantity" (1896):

> From Zeno onwards, the difficulties of continua have been felt by philosophers, and evaded, with ever subtler analysis, by mathematicians. But it seemed worthwhile to ... show, what mathematicians are in danger of forgetting, that philosophical antinomies, in this sphere, find their counterpart in mathematical fallacies. These fallacies seem, to me at least, to pervade the Calculus, and even the more elaborate machinery of Cantor's collections (*Mengen*).

Russell described Cantor's infinite ordinals as "impossible and self-contradictory", adding that the notion of a continuum is likewise contradictory.

Russell returned to Cantor's work while reviewing *De l'Infini mathématique* by Louis Couturat. Unlike most philosophers of his day, Couturat (1896) argued for the existence of both the actual infinite and infinite numbers. Regarding the notion of continuum as contradictory, Russell rejected the mathematical infinite, though more tentatively than he had done in his review of Hannequin (Russell 1897a, pp. 117, 119).

Leibniz and the antinomy of infinite number

In early 1899, Russell gave a course of lectures at Cambridge on the philosophy of Leibniz. On 24 March 1900 he wrote to Couturat that he had just finished his book on Leibniz, based on those lectures. The central contention of the book was that the Leibniz's philosophy was founded on his logic.

Three antinomies played a role in the book. The first arose from Leibniz's concept of the continuum (Russell 1900, p. 98), while the second involved the notion of cause, and thus harked back to Kant's third antinomy. Russell took this antinomy of causation very seriously, believing that it showed the inadequacy of all existing theories of dynamics (1900, p. 98). He expressed this view *after* he had ceased being a Hegelian for more than a year. Yet it is best regarded as a Hegelian residue in his philosophy.

The third antinomy that Russell discussed vis-à-vis Leibniz was the antinomy of infinite number. Leibniz, as Russell was aware (1900, pp. 108, 244), affirmed the existence of the actual infinite but denied the existence of infinite number. Indeed, at this time Russell regarded Leibniz's arguments against infinite number as "very solid" (1900, p. 109), and added: "The principle, which Leibniz also held, that infinite aggregates have no number ... is perhaps one of the best ways of escaping from the antinomy of infinite number" (1900, p. 117). This antinomy of infinite number eventually served Russell as a stepping-stone to the paradox of the largest cardinal.

Early versions of the Principles

Russell's *Principles of Mathematics* (1903) was a book that, under various titles, he had been writing since 1898. An early version was his manuscript "An Analysis of Mathematical Reasoning" (1898a), where his contradiction of relativity had a prominent place:

> It will be found that one pervading contradiction occurs almost, if not quite, universally [in mathematics]. This is the contradiction of a difference between two terms, without a difference in the conceptions applicable to them. I shall call it the contradiction of relativity. This, with addition and the manifold, appear to define the realm of mathematics. (1898a, Introduction, p. 6)

By contrast to his earlier views, Russell tentatively accepted the notion of infinite number in "Analysis":

> Infinite numbers may arise, it is true, as e.g. in the extension of number itself. But if we

> confine ourselves to conceptions contained under a genus, and leave out of account the units derived from space and time, a very special and peculiar kind of conception is required to yield an infinite extension. The number of numbers can only be discussed in connection with the necessary judgments of number.... (1898a, Bk. II, Ch. I)

This acceptance, however, proved to be temporary.

The following year Russell drafted a second version of this book, renaming it "The Fundamental Ideas and Axioms of Mathematics." Here too, antinomies played a major role. According to his outline, Part II was about "Whole and Part", whose antinomies figured prominently in Chapter IV:

> *Chap. IV. Whole and Part in Connection with Classes.* A class is what is called the extension of a concept: it consists of the terms having any given relation to any given concept.... Totality here seems necessary; but if we make it so, infinite number with its contradictions becomes inevitable, being the number of concepts as of numbers. The only way to evade the contradiction is to deny the need of totality, but even this will not serve in space and time.

The most important contradiction was that involving infinite number:

> *Chapter VII. Antinomy of Infinite Number.* This arises most simply from applying the idea of a totality to numbers. There is, and is not, a number of numbers. This [and] causality are the only antinomies known to me. This one is more all-pervading.... No existing metaphysic avoids this antinomy.

The manuscript of Part IV ("Quantity") confirms that Russell attributed a fundamental role to the antinomy of infinite number:

> Mathematical ideas are almost all infected with one great contradiction. This is the contradiction of infinity. All antinomies, I believe, so far as they are valid at all, will be found reducible to the antinomy of infinite number. (1899, p. Q33)

The third version of Russell's book, and the first manuscript to be called "The Principles of Mathematics", was drafted during late 1899 and early 1900. Here the contradiction of relativity (in so far as it concerned space and time) was definitely banished, since Russell had decided in favour of absolute position; the antinomy of causality, however, remained intact (1900a, pp. ST7, O36). Even the contradiction of relativity reappeared in a new guise, as a criticism of the notions of positive and negative number (1900a, p. O25).

In this draft of 1899–1900 the antinomy of infinite number again played a central role. Russell struggled repeatedly with the question whether there is a number of numbers and whether an infinite class has a cardinal number:

> Thus when a collection is given, it must always remain a question whether or not it has a number. It is indeed common to assume that all collections have numbers, and to say of such collections as the above [points, instants, numbers] that they have an infinite number. It is a question, with which we shall continue to be occupied throughout the greater part of this work, whether such collections have no number or an infinite number. (1900a, p. N40)

He also considered the "difficulties" surrounding the notion of infinity specifically as they concerned Cantor's work:

> The mathematical theory of infinity may almost be said to begin with Cantor.... But I cannot persuade myself that his theory solves any of the philosophical difficulties of infinity, or renders the antinomy of infinite number one whit less formidable. (1900a, p. IC40)

Thus Russell was quite ambivalent at the time toward Cantor's transfinite cardinals and ordinals. His repeated arguments against Cantor have the flavour of someone who is trying to convince himself of the opposite, and in Chapter IX he wrote in seeming hope that someone would refute his arguments:

> These difficulties, which we found in Chapter V ... are merely old puzzles worded to suit transfinite numbers. I am unaware of any answer to them, and until such an answer is found, the rejection of infinite number seems unavoidable. (1900a, p. IC85)

As in his book on Leibniz, Russell accepted the existence of infinite classes but not infinite numbers: "Infinite collections are absolutely undeniable, and it will be one of our main problems to free them from the contradictions which cling to them" (1900a, p. N45).

Russell and Whitehead travelled together to the International Congress of Philosophy held in August 1900. There Russell was extremely impressed by Peano's approach to logic, and quickly adopted his methods. In September Russell invented a logic of relations (for Peano's system), and during October he composed an article (1901) on the subject. It was very favourable toward Cantor.

During November–December 1900, Russell finished Parts III–VI of the *Principles*, not writing the final version of Parts I and II until May 1902. This manuscript of 1900–02 sheds much light on how the paradox of the largest cardinal, and then Russell's Paradox, finally emerged.

The draft of Part V contains the earliest version of the paradox of the largest cardinal. Thus, in November 1900, Russell accepted the existence of infinite numbers and in particular of Cantor's infinite ordinals and cardinals. Section 344 of the manuscript included the following passage, which he removed before publication:

> There is a certain difficulty in regard to the number of numbers, or the number of individuals or of classes. Numbers, individuals, and classes, each form a perfectly definite class, and it will be remembered that we found a general proof, from the reflexiveness of similarity, that every class must have a [cardinal] number. Now the number of individuals must be the absolute maximum of numbers, since every other class is a proper part of this one. Hence it would seem, the numbers have a maximum. But Cantor has given two proofs (1883, p. 44; 1891, p. 77) that there is no greatest number. If these proofs be valid, there would seem to be still a contradiction. But perhaps we shall find that his proofs only apply to numbers of classes not containing all individuals.... It is essential, however, to examine this point with care, before we can pronounce infinity to be free from contradictions.

Several observations must be made about this striking passage. First, it was written shortly before 24 November (1900a, pp. IC189, 199). Second, Russell aimed here to free the actual infinite from contradiction. He no longer separated the problem of the existence of infinite classes from that of the existence of infinite number, as he had done in his 1899–1900 draft. Now he accepted that any infinite class has a cardinal number. Third, and most important, the passage contains all the ingredients for the paradox of the largest cardinal, since Cantor's second proof yielded that $2^\kappa > \kappa$ for any infinite cardinal κ. For Russell, these ingredients were closely linked to the antinomy of the largest number or, what for him was almost the same, to the question whether there exists a "the number of numbers".

Yet Russell later dated his discovery of the paradox of the largest cardinal to January 1901 (Jourdain 1913, p. 146). What accounts for this discrepancy in the date? Although it might be due to a lapse of memory, it is more plausible that during November 1900 Russell did not think he had found a *paradox*; rather, he had found a "difficulty" in Cantor's theory, but one that could be resolved on further inspection. Russell still held this view in an article composed two months later (1901, p. 101).

In November 1900 Russell was also close to finding the paradox of the largest ordinal, before he had even heard of Burali-Forti's article of 1897. After doubting Cantor's claim that every set can be well-ordered, Russell noted on p. IC192:

> But, allowing this view, the ordinals will have a perfectly definite maximum, namely that ordinal which represents the type of series formed by all terms without exception. If the collection of all terms does not form a series, it is harder to prove that there must be a maximum ordinal.

On 8 December 1900 Russell wrote to Couturat about an "error" in Cantor. Here Russell expressed the paradox of the largest cardinal more succinctly than in the November draft of the *Principles*, but still did not regard it *as a paradox*:

> I have discovered an error in Cantor, who maintains that there is no largest cardinal number. But the number of classes is the largest number. The best of Cantor's proofs to the contrary can be found in [Cantor 1891]. In effect, it amounts to showing that, if u is a class whose [cardinal] number is α, the number of classes included in u (which is 2^α), is larger than α. The proof presupposes that there are classes included in u which are not individuals [members] of u; but if u = Class, that is false: [for] every class of classes is a class.[2]

At last, Russell's concern with antinomies and with seeking contradictions in both mathematical and philosophical arguments had resulted in one of the classical paradoxes of set theory. But at the time Russell did not believe that Cantor's theory was endangered. The mistake, Russell held, was simply in Cantor's denial of the existence of a largest cardinal.

Thus is was not a *new* discovery, but a *shift* in how he perceived an argument he already possessed, that later led Russell to the paradox of the largest cardinal.

[2] This passage was first pointed out by Coffa (1979).

Although Russell dated his discovery of Russell's Paradox as May 1901, the stage was set by November 1900.

Russell's Paradox emerges as a paradox

In May 1901 Russell drafted Part I of the *Principles*. On pages 22–3 in Chapter III is the first extant version of Russell's Paradox. This version, phrased in terms of predicates that are not predicable of themselves rather than in terms of classes that are not members of themselves, refuted the Principle of Comprehension:

> The axiom that all referents with respect to a given relation form a class seems, however, to require some limitation.... We saw that some predicates can be predicated of themselves. Consider now those (and they are the vast majority) of which this is not the case. These are the referents (and also the relata) in a certain complex relation, namely the combination of non-predicability with identity. But there is no predicate which attaches to all of them and to no other terms. For this predicate will either be predicable or not predicable of itself. If it is predicable of itself, it is one of those referents by relation to which it was defined, and therefore, in virtue of their definition, it is not predicable of itself. Conversely, if it is not predicable of itself, then again it is one of the said referents, of all of which (by hypothesis) it is predicable, and therefore again it is predicable of itself. This is a contradiction, which shows that all the referents considered have no common predicate, and therefore do not form a class....
>
> It follows from the above that not every definable collection of terms forms a class defined by a common predicate.

This passage, which he wrote about 15 May, shows that Russell had discovered Russell's Paradox, *as a paradox*, by May of 1901.[3]

When Russell found the paradox that now bears his name, he did not at first recognize its importance. So far as is known, he informed no one about it at the time. No reference to it occurs in his voluminous correspondence with his wife Alys, or with Couturat, before mid-1902.

Why did Russell remain silent about his paradox for an entire year? Two letters, one to Alys and one to Couturat, clarify this silence. On 25 June 1902 Russell wrote to Alys: "I have heard from Frege, a most candid letter; he says that my conundrum makes not only his Arithmetic, but all possible Arithmetics, totter" (Spadoni 1978, pp. 29–30). Here Russell referred to Frege's reply to the letter of 16 June in which Russell had first informed Frege of Russell's Paradox. What is striking about the letter to Frege is how Russell's Paradox is stated almost by happenstance. The letter gives no evidence that he regarded his paradox as essentially different from the various antinomies that he had been proposing for six years. What his letter to Alys reveals is the impact of Frege's letter on Russell. The fact that Frege, whose logical work Russell admired intensely, found Russell's Paradox devastating helped to convince him of its fundamental importance. Over the next two months they exchanged several letters about the paradox and its possible solutions (Frege 1976, pp. 211–27), causing Russell to thoroughly revise Chapter X on his paradox in the *Principles*.

The second letter to clarify Russell's year-long silence contains his first mention

[3] This was pointed out by Garciadiego (1983, p. 157).

of the paradox to Couturat. On 29 September 1902 Russell wrote that he was very busy with the *Principles*:

> I do not know what to do about a class of contradictions of which the simplest is this:
>
> $$w = x \ni (x \sim \varepsilon\, x) \supset : x\, \varepsilon\, w \,.\equiv.\, x \sim \varepsilon\, x : \supset : w\, \varepsilon\, w \,.\equiv.\, w \sim \varepsilon\, w\,.$$
>
> I have tried many solutions without success. One obtains contradictions of this sort by taking Cantor's proof that there is no largest cardinal and applying it to the class of all individuals, or of all propositions, or of all relations. When my book began to be printed, I believed that I could avoid these contradictions, but now I see that I was mistaken, a fact that greatly diminishes the value of my book. (Russell in Moore and Garciadiego 1981, p. 329)

This passage indicates that, little by little, Russell came to place his paradox at the center of his foundational concerns. So long as he believed that the paradox could be solved without great difficulty, it was not fundamental. Only after failing at many attempts to resolve it, and only after Frege underlined its significance, did Russell come to regard the paradox as crucial.

Conclusion

With few exceptions, writers on Russell have ignored, or been ignorant of, his involvement with paradoxes prior to 1901. If Russell's earlier concern with paradoxes is discussed at all, it is with perplexity. Thus G.D. Bowne wrote (1966, p. 22) about Russell's *Essay on the Foundations of Geometry*: "Russell held that the concept of a mathematical point was self-contradictory, although it is difficult to understand what 'self-contradiction' could have meant to him within the logical framework just presented."

In fact, there is a direct and continuous connection leading from Russell's early philosophical antinomies (which were Kantian or Hegelian in their inspiration), to the antinomy of infinite number, to the paradox of the largest cardinal, and then to Russell's Paradox.

In a letter of 8 February 1913 to Ottoline Morrell, Russell observed: "In matters of *work* my life has had very great continuity and unconscious unity." One such unity was Russell's pursuit of antinomies, paradoxes, and contradictions—in both philosophy and mathematics. What changed when Russell discovered Russell's Paradox in 1901, wrote about it to Frege in 1902, and published it in 1903 was that he finally, after many years, succeeded in finding a paradox so compelling that he induced the rest of world to try to solve it.

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